Chapter 3

Assignment 3 Solutions

3.1. Find an expression that expresses α (in units of dB/km) to α_p (in units of m^{-1}).

Solution: We know that

$$\frac{P(z)}{P(0)} = e^{-\alpha_p z}\,. \tag{3.1}$$

We also have

$$\begin{aligned}
\alpha &= -\frac{10}{z[\mathrm{km}]}\log\left(\frac{P(z)}{P(0)}\right) = -\frac{10}{z[\mathrm{km}]}\log\left(e^{-\alpha_p z[\mathrm{m}]}\right) \\
&= -\frac{10}{z[\mathrm{km}]}\left(-\alpha_p z[\mathrm{m}]\right)\log(e) = -\frac{10}{z[\mathrm{km}]}\left(-\alpha_p z[\mathrm{m}]\right)(0.434) \\
&= -\frac{10}{z[\mathrm{km}]}\left(-\alpha_p 1000 z[\mathrm{km}]\right)(0.434) = 4.34\times 10^3\alpha_p\,.
\end{aligned}$$

Here, we have used the transformation, $z[\mathrm{m}] = 1000\times z[\mathrm{km}]$ to change the units of z from meters to kilometers.

3.2. An optical fiber has a loss of 1 dB/km. Find the corresponding value of the loss coefficient α_p (in units of m^{-1}).

Solution: From the prior problem, we have

$$\begin{aligned}
\alpha &= 1 = 4.34\times 10^3\alpha_p \qquad (3.2)\\
\alpha_p &= \frac{1}{4.34\times 10^3} = 2.30\times 10^{-4}\mathrm{m}^{-1}\,.
\end{aligned}$$

3.3. The optical power after propagating through a fiber that is 450 m long is reduced to 30% of its original value. Calculate the fiber loss α in dB/km.

Solution: We know that $P(z)/P(0) = 0.300$ after 0.450 km of fiber, so

$$\alpha = -\frac{10}{z}\log\left(\frac{P(z)}{P(0)}\right) = -\frac{10}{0.450}\log(0.30) = 11.62\ \mathrm{dB\cdot km^{-1}}. \tag{3.3}$$

3.4. Consider an OTDR that injects a laser pulse with a pulse width of τ seconds into the test fiber at time $t = 0$.

a. Show that, at time $t = t_1$, the light returning to the input was backscattered from a segment of the fiber that is Δz long, $\Delta z = (c/n_1)(\tau/2)$

b. Show that, at $t = t_1$, the edge of the fiber segment (of size Δz) furthest from the input is a distance of $(c/n_1)(t_1/2)$ away from the input.

c. The size of Δz is the *resolution* of the OTDR. Calculate the resolution if the pulse width is 50 ns.

d. Calculate the pulse width required to have an OTDR resolution of 1 m.

e. Calculate the round-trip time for light that is backscattered from a fiber segment located 10 km from the input of the test fiber.

Solution: a) The OTDR clock starts as the leading edge of the optical pulse passes an arbitrary point at the input. If the leading edge of the pulse returns at some time t_1, then the round-trip distance is

$$\begin{aligned} 2L_1 &= \frac{ct_1}{n_1} \\ L_1 &= \frac{ct_1}{2n_1} \end{aligned} \tag{3.4}$$

where L_1 is the one-way distance down the fiber traveled by the leading edge of the pulse. If the trailing edge of the pulse is detected at time t_1, the leading edge of the pulse must have passed by τ seconds earlier (i.e., at $t = t_1 - \tau$). The distance for this segment is

$$\begin{aligned} 2L_2 &= \frac{c(t_1-\tau)}{n_1} \\ L_2 &= \frac{c(t_1-\tau)}{2n_1} \end{aligned} \tag{3.5}$$

where L_2 is the one-way distance traveled by the leading edge for this second case. At time $t = t_1$ scattered light is received from all regions of the fiber in between these extremes, i.e., from a region of length between L_1 and L_2

$$\Delta z = L_1 - L_2 = \frac{c(t_1-\tau)}{2n_1} - \frac{ct_1}{2n_1} = \left(\frac{c}{n_1}\right)\left(\frac{\tau}{2}\right). \tag{3.6}$$

b) We seen in the previous part that the furthest boundary of the region is a distance L_1 away, where $L_1 = ct_1/2n_1$.
c) The resolution of the OTDR with a 50 ns pulse is

$$\Delta z = \frac{c\tau}{2n_1} = \frac{(3.0 \times 10^8)(50 \times 10^{-9})}{(2)(1.5)} = 5 \text{ m}\,. \tag{3.7}$$

d) To achieve a resolution of 1 m, the required pulse width is found from

$$\begin{aligned} \Delta z &= \frac{c\tau}{2n_1} \\ \tau &= \frac{2n_1\,\Delta z}{c} = \frac{(2)(1.5)(1)}{3.0 \times 10^8} = 1.00 \times 10^{-8} \text{ s} = 10 \text{ ns}\,. \end{aligned}$$

e) The round-trip time is t_1, found from

$$L_1 = \frac{ct_1}{2n_1} \tag{3.8}$$

or

$$t_1 = \frac{2n_1 L_1}{c} = \frac{(2)(1.5)(10 \times 10^3)}{3.0 \times 10^8} = 1.00 \times 10^{-4} \text{ s} = 100\ \mu\text{s}\,. \tag{3.9}$$

3.5. Consider a fiber with an attenuation coefficient of α (per unit length) that has a rectangular pulse of power P_i and width τ coupled into it. The scattered light power from a segment of the fiber that is Δz long is given by $P(z)\alpha_s\,\Delta z$, where $P(z)$ is the incident light power at the location of the fiber segment and α_s is the scattering loss coefficient for material ($\alpha_s \approx 0.7$ km^{-1} for glass fibers). Not all of the scattered light, however, is guided by the fiber back to the input end of the test fiber; only a fraction, S, is. The value of S depends on the fiber properties as [Danielson, 1985]

$$S = \left(\frac{3}{16}\right)\left(\frac{\text{NA}^2}{n_1^2}\right)\left(\frac{g}{g+1}\right) \tag{3.10}$$

for multimode fibers and

$$S = \left(\frac{3}{16}\right)\left(\frac{2\lambda}{\pi n_1\,\text{MFD}}\right) \tag{3.11}$$

for single-mode fibers, where NA is the fiber numerical aperture, g is the fiber profile parameter, and MFD is the mode-field diameter. Hence, the backscattered light that is collected by the fiber is $SP(z)\alpha_s\,\Delta z$.

If Δz is given by the expression given in the previous problem, show that the backscattered power arriving back at the input at time t_1 is given by

$$P(t_1) = P_i S\alpha_s \left(\frac{\tau}{2}\right)\left(\frac{c}{n_1}\right) e^{-\alpha_p (c/n_1) t_1}\,. \tag{3.12}$$

(Note that there is a tradeoff between the power returned to the input and the resolution of the OTDR. To minimize the resolution, we want to shorten the pulse width τ; this reduces the backscattered power returned to the OTDR detector.)

Solution: The power returned at time t_1 is scattered back from a segment that is Δz long with a leading edge located at L_1, as discussed in the previous problem. This power is

$$P(t_1) = P_s e^{-\alpha_p L_1}, \tag{3.13}$$

where P_s is the scattered power, given by

$$P_s = P_i(L_1) S\alpha_s \, \Delta z, \tag{3.14}$$

as described in the problem statement. Using

$$P(L_1) = P_i e^{-\alpha_p L_1} \tag{3.15}$$

and Eqs. 3.13 and 3.14, we have

$$P(t_1) = \left(\left[P_i e^{-\alpha_p L_1}\right] S\alpha_s \, \Delta z\right) e^{-\alpha_p L_1} = P_i S\alpha_s \, \Delta z e^{-2\alpha_p L_1}. \tag{3.16}$$

Substituting in the equations for Δz and L_1 that were found in the previous problem, we have

$$P(t_1) = P_i S\alpha_s \left(\frac{c\tau}{2n_1}\right) e^{-2\alpha_p (ct_1/2n_1)} = P_i S\alpha_s \left(\frac{c\tau}{2n_1}\right) e^{-\alpha_p (ct_1/n_1)}. \tag{3.17}$$

This is the desired result.

3.6. Consider a step-index multimode fiber with $\mathrm{NA} = 0.2$, $n_1 = 1.5$, $\alpha_{\text{fiber}} = 0.6$ dB·km^{-1}, and $\tau = 50$ ns.

a. Using the results of the previous problem, calculate the ratio of the backscattered power to the input power for $t_1 = 0$ (i.e., for light backscattered from the segment of the fiber located right at the input of the test fiber).

b. Calculate the ratio of the backscattered power to the input power for a segment of fiber located 10 km from the input of the fiber.

c. Calculate the ratio of the backscattered power from the segment located 10 km from the input to the backscattered power from the segment located right in front of the input.

d. The dynamic range, DR, (in dB) of an OTDR is defined as

$$\mathrm{DR} = 10 \log\left(\frac{P_{\mathrm{s\ max}}}{P_{\mathrm{s\ min}}}\right), \tag{3.18}$$

where $P_{\mathrm{s\ max}}$ is the power of the strongest backscattered signal that can be detected at the OTDR receiver and $P_{\mathrm{s\ min}}$ is the power of the weakest signal that can be detected. If the dynamic range of the detector is 20 dB, calculate the maximum operating range of the OTDR, assuming that fiber loss is the only loss encountered (i.e., there are no connector or splice losses). You may assume that the coupler that connects the laser to the OTDR fiber pigtail (and the detector to the pigtail) has a 3 dB loss for each pass through it.

Solution: We first calculate S for this fiber (recognizing that $g = \infty$ for a step-index fiber). **Revised 4/29/99**

$$S = \frac{3}{16}\frac{\mathrm{NA}^2}{n_1^2}\left.\frac{g}{g+1}\right|_{g=\infty} = \frac{3}{16}\frac{\mathrm{NA}^2}{n_1^2} = \left(\frac{3}{16}\right)\left(\frac{0.2^2}{1.5^2}\right) = 3.33\times 10^{-3}\,. \tag{3.19}$$

Only 0.333% of the light is backscattered.

We will also need to convert the fiber loss from α [dB/km] to α_p [1/m].

$$\alpha_p = \frac{\alpha[\mathrm{dB/km}]}{4.35\times 10^3} = \frac{0.6}{4.35\times 10^3} = 1.379\times 10^{-4}\ \mathrm{m}^{-1}\,. \tag{3.20}$$

a. At $t_1 = 0$

$$\begin{aligned} P(0) &= P_i S\alpha_s\left(\frac{c\tau}{2n_1}\right) \qquad (3.21)\\ \frac{P(0)}{P_i} &= S\alpha_s\left(\frac{c\tau}{2n_1}\right)\\ &= (3.33\times 10^{-3})(7\times 10^{-4})\left(\frac{(3.0\times 10^8)(50\times 10^{-9})}{(2)(1.5)}\right)\\ &= 1.167\times 10^{-5} \Rightarrow -49.3\ \mathrm{dB}\,. \end{aligned}$$

b. At a distance of 10 km, $L_2 = 10\times 10^3$ and

$$t_1 = \frac{2n_1L_2}{c} = \frac{(2)(1.5)(10\times 10^3)}{3.0\times 10^8} = 1.000\times 10^{-4}\ \mathrm{s}\,. \tag{3.22}$$

The power ratio, $P(t_1)/P_i$ is

$$\begin{aligned} \frac{P(t_1)}{P_i} &= S\alpha_s\left(\frac{c\tau}{2n_1}\right)e^{-\alpha_p(ct_1/n_1)}\\ &= (3.33\times 10^{-3})(7\times 10^{-4})\left(\frac{(3.0\times 10^8)(50\times 10^{-9})}{(2)(1.5)}\right)\\ &\qquad \times e^{-(1.379\times 10^{-4})\left((3.0\times 10^8)(1.00\times 10^{-4})/1.5\right)}\\ &= 2.94\times 10^{-6} \Rightarrow -53.6\ \mathrm{dB}\,. \end{aligned}$$

c. The ratio of the power levels, $P(t_1)/P(0)$, is

$$\frac{P(t_1)}{P(0)} = \frac{\dfrac{P(t_1)}{P_i}}{\dfrac{P(0)}{P_i}} = \frac{2.94\times 10^{-6}}{1.167\times 10^{-5}} = 0.0252 \Rightarrow -5.99\ \mathrm{dB}\,. \tag{3.23}$$

d. The dynamic range of the OTDR is specified to be 20 dB ($\Rightarrow$ 100).

$$\mathrm{DR} = \frac{P_{\mathrm{s\ max}}}{P_{\mathrm{s\ min}}} = \frac{P_{\mathrm{s\ max}}}{P_i}\frac{P_i}{P_{\mathrm{s\ min}}} = \frac{\dfrac{P(0)}{P_i}}{\dfrac{P(t_1)}{P_i}} = e^{\alpha_p(2L_2)} \tag{3.24}$$

or

$$\begin{aligned} L_2 &= -\frac{\ln(1/\mathrm{DR})}{2\alpha_p} = \frac{\ln(\mathrm{DR})}{2\alpha_p} \\ &= \frac{\ln(100)}{(2)(1.379\times10^{-4})} = 2.17\times10^{4}\ \mathrm{m} = 21.7\ \mathrm{km}\,. \end{aligned}$$

Hence, we find that this OTDR can work out to about 22 km of this fiber. Returns from a larger distance will be lost in the noise floor of the OTDR.

3.7. Consider a single-mode fiber operating at 1300 nm with a mode-field diameter of 9 μm, $n_1 = 1.5$, and $\alpha = 0.5$ dB$\cdot$km^{-1}. Assume, again, that $\tau = 50$ ns. Repeat the calculations of the previous problem.

Solution: The solutions follows the steps of the previous problem.

We first calculate S for this fiber.

$$S = \frac{3}{16}\frac{2\lambda}{\pi n_1\,\mathrm{MFD}} = \frac{3}{16}\frac{(2)(1300\times10^{-9})}{(\pi)(1.5)(9\times10^{-6})} = 1.150\times10^{-2}\,. \tag{3.25}$$

So, we find that 1.150% of the light is backscattered.

We will also need to convert the fiber loss from α to α_p.

$$\alpha_p = \frac{\alpha[\mathrm{dB/km}]}{4.35\times10^{3}} = \frac{0.5}{4.35\times10^{3}} = 1.149\times10^{-4}\ \mathrm{m}^{-1}\,. \tag{3.26}$$

a) At $t_1 = 0$

$$\begin{aligned} P(0) &= P_i S\alpha_s\left(\frac{c\tau}{2n_1}\right) \\ \frac{P(0)}{P_i} &= S\alpha_s\left(\frac{c\tau}{2n_1}\right) \\ &= (1.150\times10^{-2})(7\times10^{-4})\left(\frac{(3.0\times10^{8})(50\times10^{-9})}{(2)(1.5)}\right) \\ &= 4.02\times10^{-5} \Rightarrow -44.0\ \mathrm{dB}\,. \end{aligned} \tag{3.27}$$

b) At a distance of 10 km, $L_2 = 10\times10^{3}$ and

$$t_1 = \frac{2n_1L_2}{c} = \frac{(2)(1.5)(10\times10^{3})}{3.0\times10^{8}} = 1\times10^{-4}\ \mathrm{s}\,. \tag{3.28}$$

The power ratio is

$$\begin{aligned}
\frac{P(t_1)}{P_i} &= S\alpha_s\left(\frac{c\tau}{2n_1}\right)e^{-\alpha_p(ct_1/n_1)} \\
&= (1.150\times10^{-2})(7\times10^{-4})\left(\frac{(3.0\times10^{8})(50\times10^{-9})}{(2)(1.5)}\right) \\
&\quad \times e^{-(1.149\times10^{-4})\left((3.0\times10^{8})(1.00\times10^{-4})/1.5\right)} \\
&= 4.04\times10^{-6}\Rightarrow -53.9\text{ dB}\,.
\end{aligned}$$

c) The ratio of the power levels is

$$\frac{\frac{P(t_1)}{P_i}}{\frac{P(0)}{P_i}} = \frac{4.04\times10^{-6}}{4.02\times10^{-5}} = 0.1004 \Rightarrow -9.98\text{ dB}\,.$$

d) The dynamic range of the OTDR is specified to be 20 dB ($\Rightarrow$ 100).

$$\mathrm{DR} = \frac{P_{\mathrm{s\ max}}}{P_{\mathrm{s\ min}}} = \frac{P_{\mathrm{s\ max}}}{P_i}\frac{P_i}{P_{\mathrm{s\ min}}} = \frac{\frac{P(0)}{P_i}}{\frac{P(t_1)}{P_i}} = e^{\alpha_p(2L_2)} \tag{3.29}$$

Hence,

$$\begin{aligned}
L_2 &= -\frac{\ln(1/\mathrm{DR})}{2\alpha_p} = \frac{\ln(\mathrm{DR})}{2\alpha_p} = \frac{\ln(100)}{(2)(1.149\times10^{-4})} \\
&= 2.00\times10^{4}\text{ m} = 20.0\text{ km}\,.
\end{aligned}$$

Hence, we find that this OTDR can work out to about 20 km of this fiber. Returns from a larger distance will be lost in the noise floor of the OTDR. Hence, we see that the dynamic range of the receiver can restrict the operating range of the OTDR.

3.8. A break occurs in a fiber with a loss of 3 dB/km. The output power from an OTDR set used to locate the break is 250 mW and the detected echo power is 1 μW. Approximately 10 dB of loss are encountered in coupling the OTDR signal into the fiber and the returned signal encounters 6 dB of loss at the optical splitter (see Fig. 13.1). The reflectivity of a perpendicular break in the fiber is approximately 4%; the average reflectivity of a nonperpendicular break is about 0.5%. Using the latter value of reflectivity, calculate the distance to the break in the fiber.

Solution: The source power P_s is

$$P_s = 250\text{ mW} = +24.0\text{ dBm}\,. \tag{3.30}$$

The detected power P_d is

$$P_d = 1\ \mu\text{W} = 1 \times 10^{-3}\ \text{mW} = -30.0\ \text{dBm}\,. \tag{3.31}$$

The power that is coupled into the fiber at the source end P_1 is

$$P_1 = P_s - \eta_{\text{couple}} = 24.0 - 10 = 14\ \text{dBm}\,. \tag{3.32}$$

The power that reaches end of the fiber P_2 is

$$P_2 = P_1 - \alpha L = 14 - 3L\ \text{dBm}\,. \tag{3.33}$$

The power reflected at the end P_3 is

$$\begin{aligned} P_3 &= P_2 - \eta_{\text{reflection}} = P_2 - 10\log(0.005) \\ &= (14.0 - 3L) - 23.0 = -9.0 - 3L\ \text{dB}\,. \end{aligned} \tag{3.34}$$

The power P_4 that returns to the source end of the fiber after traversing a distance L through the fiber is

$$\begin{aligned} P_4 &= P_3 - \alpha L = P_3 - 3L \\ &= (-9.0 - 3L) - 3L = -9.0 - 6L\ \text{dBm}\,. \end{aligned} \tag{3.35}$$

The power coupled out of the fiber and to the detector P_{det} is

$$P_{\text{det}} = P_4 - \eta_{\text{couple}} = (-9.0 - 6L) - 6.0 = -15.0 - 6L\ \text{dBm}\,. \tag{3.36}$$

So

$$\begin{aligned} P_{\text{det}} &= -15.0 - 6L = -30\ \text{dBm} \\ L &= \frac{30.0 - 15.0}{6} = 2.50\ \text{km}\,. \end{aligned} \tag{3.37}$$

3.9. The Sellmeier equations give us an expression for $n(\lambda)$ based on the resonances of the absorbing atoms or molecules. The equation for silica is

$$n(\lambda) = \sqrt{1 + \sum_{k=1}^{3} \frac{G_k \lambda^2}{\lambda^2 - \lambda_k^2}}\,. \tag{3.38}$$

The values of G_k and λ_k represent the resonant wavelengths and relative strengths of the resonance and are found in the following Table.

λ_k (nm)	G_k
68.4	0.69617
116.2	0.40794
9896.2	0.89748

Values of the parameters for other materials are found in optical handbooks.

a. Using a computer, plot $n(\lambda)$ for values of λ between 600 nm and 1600 nm.

b. Using a computer, plot $(\lambda^2\, d^2n/d\lambda^2)$ for values of λ between 600 nm and 1600 nm.

Solution: The Sellmeier equations were used to plot Figs. 3.7 (p. 47) and 3.8 (p. 48). Your plots should look the same.